Combinatorics Round Solutions

May 17, 2026

LAMT 2026

1. A full house is a set of five cards consisting of three cards of one rank, along with two cards from a different rank. (Rank refers to the numerical or face value assigned to a card; there are 13 ranks: $A, 2, 3 \ldots, 10, J, Q, K$.)

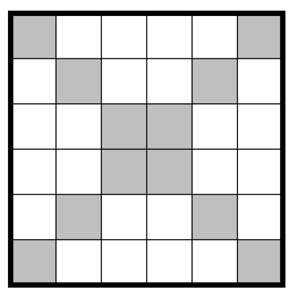
   Find the minimum number of cards that need to be drawn from a standard deck of 52 cards to guarantee that five of them will form a full house.

   Solution: $\boxed{27}$

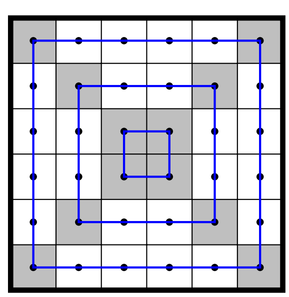
   We claim that $\boxed{27}$ is the correct answer. By Pigeonhole Principle, since $\lceil \frac{27}{13} \rceil = 3$, there will exist three of one rank. Then, there will be either 23 or 24 cards left that are not of that rank. Since $\lceil \frac{23}{12} \rceil = 2$, there will exist a pair of a different rank in both cases, which guarantees the existence of a full house.

   26 is insufficient, since we can just have a pair from each rank, which prevents the existence of a full house.

2. In a 6×6 grid, the cells on the two main diagonals are shaded. Tom starts in a cell, and moves to an adjacent cell every second. However, he may only change direction if he is in a shaded cell (he can pick his starting direction). Find the number of ways Tom can start in a cell, and follow a path which visits at least 3 distinct cells (without repeating any cells) that leads him to his starting cell (which is the only cell that can be repeated).



   Solution: $\boxed{72}$



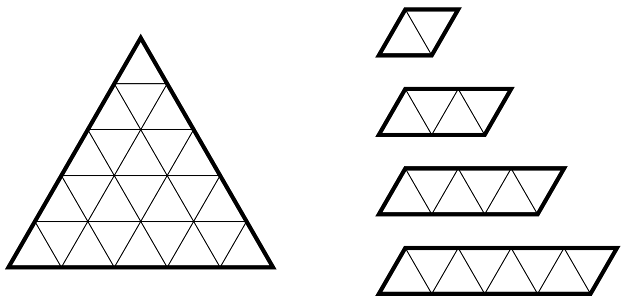
   The only paths that work are cycles shaped like a square centered at the center of the grid. There are 36 options for the starting cell, and 2 options for the direction, giving an answer of $2 \cdot 36 = \boxed{72}$.

   Suppose we had some other valid path, with starting cell S. This would involve leaving one of the three concentric squares to go to another one. We would have to travel in a straight line until we reached a shaded cell, which we denote C. Observe that the moment we leave the square containing C, we will no longer be able to change directions. It is easy to see that this forces C to be our exit cell from this square, which means cell C is repeated on our path, which is not allowed.

Remark. In the original version of the test, the wording was quite ambiguous. Specifically, the condition that a path had at least 3 cells was not mentioned, and was issued as a clarification.

Without this, the answer is 112, as there are 40 degenerate paths where we travel to a red cell in one move, and immediately turn around and return to our starting cell. This edge case was not intended, and none of the reviewers and very few students observed it. Because of this, the clarification was made during the test to avoid this edge case. Unfortunately not all students were aware of this, and we were only able to accept the intended answer of 72. We apologize to any students who were negatively affected by this error.

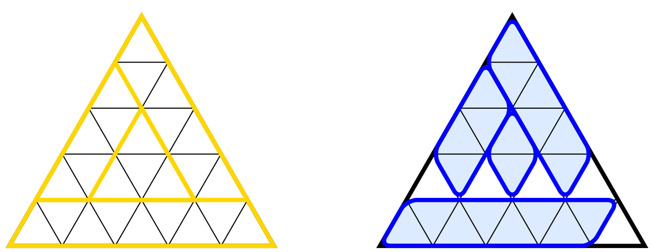
3. The equilateral triangle on the left has side length 5, and gridlines tile it into equilateral triangles of side length 1. Find the number of ways to place all 4 parallelograms pictured on the right onto the large equilateral triangle, so that their boundaries lie on the gridlines and no two of them overlap. The parallelograms may be rotated or reflected.



Solution: $\boxed{648}$

Start with the largest parallelogram, which must be placed along the outside of the triangle. By inspection, there are 6 ways to place it, as it occupies almost an entire side of the triangle, except for one small equilateral triangle of side length 1.

Once this is placed, we are left with an equilateral triangle of side length 4. The next largest parallelogram has 6 options, and leaves us with a triangle of side length 3. Then we place the 3rd largest parallelogram, with 6 options again.



Finally we have a triangle of side length 2, and there are 3 ways to place the smallest parallelogram. In total the answer is $6 \cdot 6 \cdot 6 \cdot 3 = \boxed{648}$.

4. Arpit selects two nonempty sets A and B of integers from 1 to 2026 inclusive. Suppose $\min(A) = \max(B)$, $|A| = \min(B) + 2$, and $|B| = \max(A) - 2$. Find the number of ways Arpit could have selected the two sets A and B.

Solution: $\boxed{2024}$

The key facts to use are

$$|A| \le \max(A), \qquad \min(A) \le \max(A).$$

Using this, we have

$$\begin{aligned}\max(A) - 2 &= |B| \\ &\le \max(B) \\ &= \min(A).\end{aligned}$$

So $\max(A)$ and $\min(A)$ are at most 2 apart, meaning A has at most 3 elements. But we also have

$$|A| = \min(B) + 2 \implies |A| \geq 3,$$

so we must have $|A| = 3$. Then let $\max(B) = \min(A) = n$. The above equation $\min(B) = 1$. We also have $A = \{n, n+1, n+2\}$, so $|B| = n$, which forces

$$B = \{1, 2, \dots, n\}.$$

Given a choice of n, we can get unique choices for B and A. Since $n + 2 \leq 2026$, n can range from 1 to 2024, so the answer is $\boxed{2024}$.

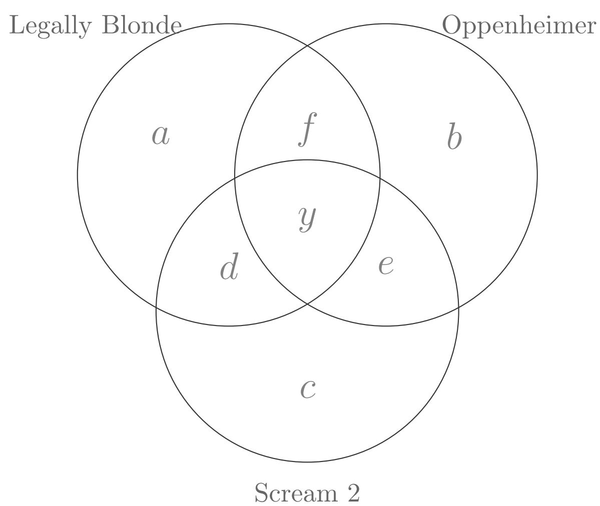
Set B must be 1 to k and set A must be k, $k+1$, $k+2$.

5. Vicky conducts a survey, asking UCLA students about their opinions on the movies Oppenheimer (2023), Legally Blonde (2001), and Scream 2 (1997). She notes the following results:

   - Among the students who liked both Oppenheimer and Legally Blonde, one-third also liked Scream 2.
   - Among the students who liked Scream 2, one-sixth also liked at least one other movie listed.
   - Among the students who liked less than 3 of the movies listed, one-sixth liked two of the movies listed.
   - Being UCLA students, everyone polled liked at least one of the movies listed.

   At random, Vicky chooses someone who did not like Scream 2. Find the probability they like exactly one of the three movies listed above.

   Solution: $\boxed{\frac{5}{7}}$



   From the statements, we can conclude (using the Venn diagram labeling) $f = 2y$, $c = 5(d + e + y)$, and $a + b + c = 5(d + e + f)$. Plugging the first and second equality into the last, we have $a + b + 5(d + e + y) = 5(d + e + 2y) \implies a + b = 5y$.

   We want the value of $\frac{a+b}{a+b+f} = \frac{5y}{5y+2y} = \boxed{\frac{5}{7}}$.

6. Kenneth starts at the point $(0, 0)$ in the coordinate plane. Every second, he chooses to move right by some integer power of 2, or up by some integer power of 2. Suppose he never goes above the line $y = x$, and never uses a power of 2 more than once. Find the number of ways Kenneth can reach the point $(73, 54)$.

   Solution: $\boxed{1120}$

Writing 73 and 54 in binary we get 1001001 and 011010. We require jumps of length 64, 8, and 1 to be made right, and jumps of length 32, 16, 4, and 2 to be made up. The condition of $y \leq x$ is equivalent to having the largest power of 2 we've used be 64, 8 or 1, at any given moment.

Note that the 1 doesn't interact with anything, so there are 7 ways to place it. Then we require 64 or 8 to be first.

Case 1: 64 is first. Then the remaining numbers can be permuted in 5! ways.

Case 2: 8 is first. Then we require 64 to appear before 32 and 16, so there are $\frac{5!}{3} = 40$ ways to do this.

In total our answer is $7 \cdot (120 + 40) = \boxed{1120}$.

7. Nish has 20 boxes of cookies, having 1 cookie, 2 cookies, all the way to 20 cookies in the 20th box. If two boxes have a and b cookies, Nish can combine them into a single box of $a + b$ cookies in ab seconds. Find the minimum number of seconds it takes for Nish to combine all the cookies into one box.

   Solution: $\boxed{20615}$

   We can imagine a graph connecting cookies, where cookies in the same box are connected. Note that when we combine two boxes with a and b cookies in them we form ab new edges. Thus it takes 1 second to form a single edge. Thus the answer is the total number of edges we need to form. There are 210 total cookies, so the answer is

$$\binom{210}{2} - \left(\binom{20}{2} + \binom{19}{2} + \cdots + \binom{2}{2}\right) = \binom{210}{2} - \binom{21}{3} = \boxed{20615}.$$

8. Find the number of ways to partition the integers from 0 through 41 into 14 disjoint sets of 3 integers such that each set is of the form

$$\{n, n+1, n+22\} \text{ or } \{n, n+21, n+22\}$$

   for some integer n. (All elements in the set are taken modulo 42)

   Solution: $\boxed{384}$

   Consider a 2×21 grid filled with the numbers from 1 through 42 in order. Then note that each set is an L-shaped tromino of cells (potentially looping around the end of the grid). So it suffices to compute the number of ways to complete this tiling.

   Two trominoes must be placed together to form a 2×3 grid, and there are 2 ways to create this. There are three ways to decide where each of the seven 2×3 grids starts and ends, giving us an answer of $3 \cdot 2^7 = \boxed{384}$.

   Remark: There are easier ways to think about this, but the problem was written backwards by starting with this setup.

9. In a tournament with 6 players, every pair of players plays exactly one game, and each game has exactly one winner. The tournament is called *boring* if there are exactly three ordered triples of distinct players (A, B, C) such that A beats B, B beats C, and C beats A. Find the number of ways to assign a winner to each game, so that the tournament is boring.

   Solution: $\boxed{960}$

   Each triple can have the elements cycled in 3 ways:

$$(A, B, C) \iff (B, C, A) \iff (C, A, B).$$

   So three ordered triples is equivalent to saying that there is exactly one directed cycle in the graph.

   Suppose the unique directed 3-cycle involves players $\{a, b, c\}$, say $a \to b \to c \to a$. There are $\binom{6}{3}$ ways to choose these players, and then 2 ways to determine the direction of the cycle, so multiply by 40 at the end.

   Consider any other player $x \notin \{a, b, c\}$. We claim x must beat all three of $\{a, b, c\}$ or lose to all three.

Suppose x beats exactly two of them, say x beats a and b but loses to c. Then in the triple $\{x, b, c\}$: $x \to b$, $b \to c$, $c \to x$, forming a new directed 3-cycle. This contradicts uniqueness. Similarly, if x beats exactly one of $\{a, b, c\}$, a new 3-cycle arises. In all cases, uniqueness fails.

Now we take cases on the behavior of the other 3 players, d, e, and f. Call a player a "loser" if they lose to all of a, b, c, and a winner if they win against all of them.

- There are 3 losers. It's easy to see that the only way to fail is if d, e, and f form a directed cycle. This gives $2^3 - 2 = 6$ cases.
- There are 2 losers. There are 3 ways to select who is the winner among the three. Suppose d is the winner. Looking at the graph, the edges de and df are both forced (d must win). Then the edge ef can be anything. This gives $3 \cdot 2 = 6$ cases.

Note that 1 loser is symmetric to 1 winner (2 losers), and same with 3 and 0 losers. So multiply these cases by 2 to get $2 \cdot (6 + 6) = 24$. Multiplying by the 40 from earlier, the answer is $40 \cdot 24 = \boxed{960}$.

10. Ricky starts the bottom left cell of an 8×8 grid, and every second he can move to the cell either above or to the right of his current cell. There are 8 boulders placed in distinct cells according to following rules:

- If a cell has a boulder, then the cell diagonally adjacent that is to the bottom-right of this cell is empty.
- Ricky may not enter a cell with a boulder.
- The bottom left cell and top right cell do not contain boulders.

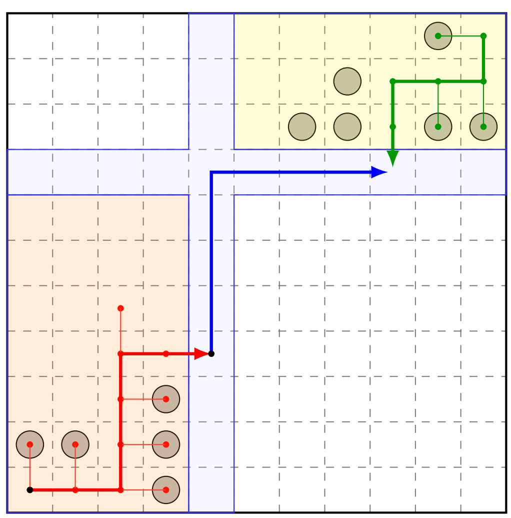
Find the number of arrangements of boulders for which Ricky cannot reach the top right cell of the grid.

Solution: $\boxed{2574}$

For 8 replaced by n, the answer is more generally $2\binom{2n-3}{n}$.

Claim: Define an "empty" row or column to be a row or column with no boulders. Then there cannot both be an empty row and an empty column.

Proof. Suppose otherwise. If we are currently in the cell (x, y), then at least one of the cells $(x + 1, y)$ and $(x, y + 1)$ must be empty, so we can travel into it (unless we are on the edge of the grid).

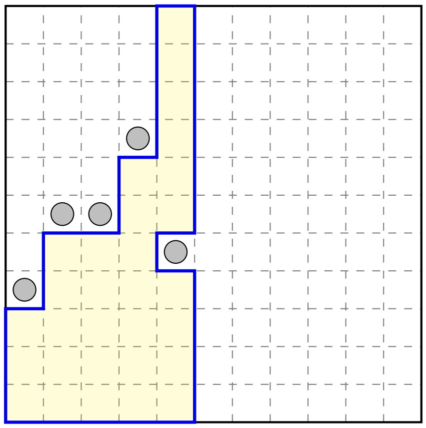


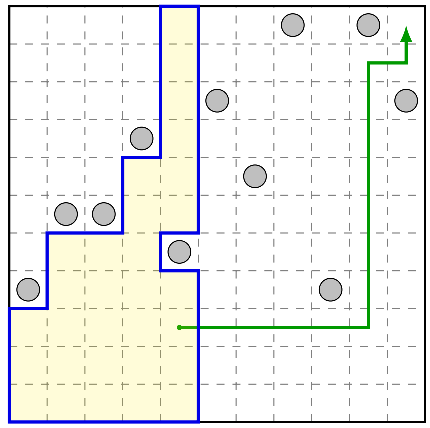
Then using this, we can find a path into the empty row or column from the top right cell as well as the top left cell. Then connecting these gives a valid path. □

Without loss of generality assume that there is no empty column. This means each column must contain exactly one boulder. Let these boulders be $(1, a_1)$, $(2, a_2)$, $\ldots$, (n, a_n).

Claim: We must have $a_i \geq a_{i-1}$ for all $1 < i \leq n$.

Proof. Suppose otherwise. Then there exists some $i \leq n$ for which $a_i \leq a_{i-1} - 2$, since a_i cannot equal $a_{i-1} - 1$. Consider the minimal such i. Then for all $j < i$, the cells below the boulders (j, a_j) are reachable. Additionally $(i, a_i + 1)$ is reachable from $(i - 1, a_i + 1)$, since $a_i + 1 < a_{i-1}$. From this, we see that all cells above (i, a_i) and below (i, a_i) are reachable. All these reachable cells are pictured in yellow below.



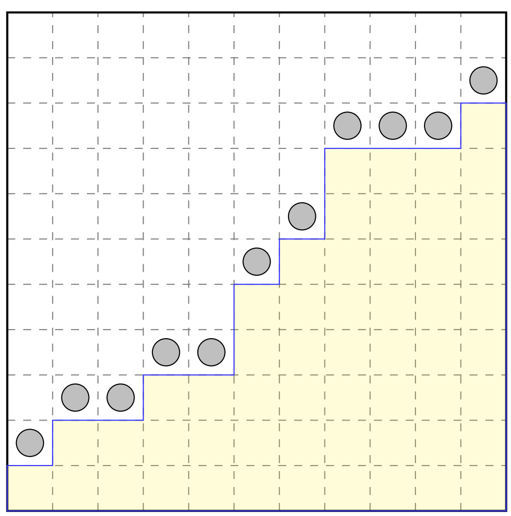


Now consider the $n - i$ columns to the right of the ith column. If the top row in these columns is empty, then we can travel to the top row and complete our path.

Otherwise, note that $n - i < n - 1$, so by Pigeonhole there exists an empty row in this $n - i \times n$ grid which we can reach. Then travel along this row until we have the same x coordinate as the rightmost boulder in the top row. Travel up to that boulder, then go to its right to complete the path. $\square$

Next, note that the cases where each row contains a boulder and each column contains a boulder are disjoint. Thus only focus on the case where every column contains a boulder.

The top row and column must be empty, as this would imply the existence of a boulder in the bottom left or top right cell. Then it suffices to compute the number of ways to place n boulders in $n - 2$ rows so that their y coordinates are nondecreasing.



We can think of this as a path from $(0, 1)$ to $(n, n - 2)$, placing a boulder on top of each horizontal segment of our path. This gives us $\binom{2n-3}{n}$ paths, making our final answer

$$2 \cdot \binom{2 \cdot 8 - 3}{8} = 2 \cdot \binom{13}{8} = \boxed{2574}.$$

11. **[TIEBREAKER]** A polyomino is a plane geometric figure formed by joining unit squares edge-to-edge. Two polyominoes are considered the same if one can be obtained from the other by rotation (but not reflection). For example:

- There is 1 domino (2 squares)
- There are 2 trominoes (3 squares)
- There are 7 tetrominoes (4 squares, which are the Tetris pieces).

Estimate the number of unique decominoes (polyominoes made of 10 unit squares), where rotations are considered identical but reflections are not.

Express your answer as a number in base 10 (submissions not in this form will not be accepted). Ties will be broken based on distance to the correct answer.

Solution: $\boxed{9189}$